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ABSTRACT

This document presents four research papers related to the role of inconsistent ideas in learning mathematics and a discussion paper related to the same issue. The titles of the papers are: (1) "Cognitive Conflict in Procedure Applications" (Merlyn Behr and Guershon Harel); (2) "Inconsistencies in Preservice Elementary Teachers' Beliefs and about Multiplication and Division" (Anna O. Graeber and Dina Tirosh); (3) "The Relationship of Students' Definitions and Example Choices in Geometry" (Patricia S. Wilson); and (4) "Inconsistencies in the Learning of Calculus and Analysis" (David Tall). The discussion paper is entitled "Cognitive Conflict in Learning Mathematics" (Leslie P. Steffe). (PK)

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THE ROLE OF INCONSISTENT IDEAS IN LEARNING MATHEMATICS

American Educational Research Association

Annual Meeting

April 7, 1988; New Orleans, LA

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THE ROLE OF INCONSISTENT IDEAS IN LEARNING MATHEMATICS

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Sponsored by SIG/Research in Mathematics Education

The Role of Inconsistent Ideas in Learning Mathematics is a topic that needs increased attention. I am grateful to the researchers and discussants who have given time and concentrated effort to develop a symposium addressing inconsistencies. I am especially grateful to Jim Kaput who agreed to present David Tall's paper with short notice.

We hope to stimulate an interest in this area of cognitive research. I encourage you to correspond with the participants and to share your work. Addresses are listed in the back of this booklet.

Patricia S. Wilson
Symposium Organizer

The Role of Inconsistent Ideas in Learning Mathematics

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Cognitive Conflict in Procedure Applications

Merlyn Behr and Guershon Harel
Northern Illinois University

It has been shown that students' errors and misconceptions in different fields of mathematics and science do not occur haphazardly, but that there is a uniformity and pattern in the students' erroneous behavior. Efforts have been made by researchers to formulate a theory that would account for systematic errors in mathematics; such as made by Matz (1982) in the domain of high-school algebra problem solving and by Brown and VanLehn (1982) in the domain of place-value subtraction. The benefit of such a theory for teaching is that an understanding of the origin of the sources of students' errors might help in designing an appropriate learning environments so that the errors can be inhibited. This paper is an attempt to contribute to the formulation of that theory by offering a global account for some observed systematic errors in different domain of mathematics.

Brown and VanLehn (1982) indicated that an error in solving a problem is frequently the result of a repair to a known rule or procedure that a student invents and applies to circumvent a cognitive conflict which results from differences between his or her existing knowledge and the constraints of the problem space. In the first section of this paper we suggest a classification of some situations that students face which result in cognitive conflict. We indicate possible sources of these conflicts, and strategies which students use to resolve, remove, or circumvent these conflicts. Conflict resolution, as we will see, often results in some application of an erroneous procedure. In the second section of the paper we will offer a global account for some observed systematic errors in the domain of decimal numerations, algebra, and proportional reasoning, which result from one of the conflict types we have identified. This account is based on the hypothesized existence of a global problem solving rule, called the Matching Rule. We first observed this rule in a non-numeric proportional reasoning task with 7-th grade children (Harel, Behr, and Post, in preparation).

The hypothesized existence of the Matching Rule provides insight into how students perceive similarity between problem structures. It explores the information processing that

students employ in determining similarity between problem structures in order to apply a known procedure to solve an unfamiliar problem.

Using the Matching Rule, we offer an alternative explanation of the students' errors in algebra from that suggested by Matz (1982). Matz hypothesized three schemata to account for one of the largest and most frequently occurring class of errors typified by $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$, $(x+y)^2=x^2+y^2$, generalized distribution over addition. We, on the other hand, hypothesize the existence of a single global rule, the Matching Rule, which governs the errors described by Matz and others that students make in other domain of mathematics as well.

Classification of Cognitive Conflicts

We have classified situaions which result in cognitive conflict into four categories; the first two are the same as two identified by Brown and VanLehn (1982).

Missing a fragment . In this situation the student is missing a subprocedure of a correct procedure, either because it was never learned or because it has been forgotten. In this situation the student may choose one of two avenues to circumvent the conflict, quit or invent a subprocedure (usually erroneous).

Violation of a rule. Here, the student applies a procedure that results in a violation of a previouly acquired rule. In this case the student may (a) invent a new procedure or subprocedure, or (b) reinterpret the problem so that it conforms to a rule which he or she currently possesses. Brown and VanLehn (1982) identified this conflict in the domain of place-value subtraction. An example from the domain of multiplication word problems is that a multiplier less than one violates the intuitive rule of "multiplication makes bigger," which students acquire previously in the whole number domain (Bell, Swan, and Taylor, 1981; Fischbien, Deri and Nello, 1985). One response to this constraint violation is that the student reinterprets the problem situation to be of division type, because with this interpretation the solution would conform to another intuitive rule, "division makes smaller."

Unexpected result. In this case a known procedure can be applied, but it leads to an unexpected result or to a result which violates the student's concept image. An example is when a student is asked to compare the cardinalities of the set of whole numbers and the set of odd whole numbers. A cognitive conflict arises because the one-to-one correspondence procedure does not lead to the expected result; namely, that the one-to-one procedure exhausts the subset and the superset simultaneously. The rule which students learn that the cardinality of a subset must be smaller than that of its supperset is true only in the domain of finite sets. Since this rule is

learned without undrstanding of the finite set constraint, congitive conflict arises in the domain of infinite sets. Another example is that many students find it difficult to understand that the function defined by the rule f(x)= x for x rational and f(x) = -x for x irrational is continuous at x=0. The conflict is that when the student applies a correct procedure (e.g., the standard ϵ-δ-procedure) to check the continuity of the function f at x=0, it results in an answer which contradicts the concept image of a continuous function he or she acquired previously. (For an interesting discussion about concept image versus concept definition, see Vinner, 1983.) In many cases, conflicts of this type are difficult to resolve; they arise because of students' belief system for mathematics and their understanding of what a mathematical definition is.

Matching problem components. From the student's perspective, this situation is characterized by the fact that the components of a familiar problem match the components of a new problem. Consequently, the student would identify the two problems to be of the same type. Because the problems are seen by the student to be the same, application of a new or different procedure to solve the new problem would create a conflict. To circumvent this conflict the student would apply the familiar procedure to the new problem. An example is the erroneous procedure children are known to use when asked to order two decimal numbers such as 2.34 and 2.8. They state that 2.34 is greater because it has more digits (Nesher, 1986, Hiebert, 1986). It appears that students observe that the lengths of the digit strings for the pair 2.34 and 2.8 are the same as for any pair of three and two digit whole numbers, 234 and 28, for example. Moreover, 234 is known to be greater than 28 because of the "length-of-digit-string" rule. Application of this digit-string rule to 2.34 and 2.8 leads to the conclusion that 2.34 is greater than 2.8.

The Matching Rule

The Matching Rule can be characterized as follows: Given two pairs of problem components (A, B) and (C, D) and two, or more, trichotomous relations R_1 and R_2, such that the value of R_1 for A and B and for C and D is the same, and if the value of R_2 for A and B is known, then a conclusion can be drawn that the value of R_2 for C and D is the same as for A and B. Or, if the values of R_1 and R_2 for A and B are the same, and if the value of R_1 for C and D is known, then a conclusion can be drawn that the value of R_2 for C and D is the same as of R_1 for C and D. Schematically this can be stated as follows:

If $R_1(A, B)$, $R_1(C, D)$, and $R_2(A, B)$ are known, then $R_1(A, B)=R_1(C, D)$ implies $R_2(C, D)=R_2(A, B)$, or

If $R_1(A, B)$, $R_2(A, B)$, and $R_1(C, D)$ are known, then $R_2(A, B)=R_1(A, B)$ implies $R_1(C, D)= R_2(C, D)$.

The erroneous procedure children use to order two decimal numbers can be described in terms of the Matching Rule as follows: The relationship between the lengths of the digit strings, R_1, for A=2.34 and B=2.8 is observed to be the same as for any pair of three and two digit whole numbers, C=234 and D=28, for example. That is, it is observed that $R_1(A, B)$ and $R_1(C, D)$ are both the > relation; that is, $R_1(A, B)=R_1(C, D)$. Since it is known that the order relation, R_2, for C=234 and D=28 is the > relation, it is concluded that R_2 for A=2.34 and B=2.8 is also the > relation; that is, $R_2(C, D)=R_2(A, B)$. Before we give more examples of how students apply this rule to solve problems in different domains of mathematics, we will show where we originally observed this rule being applied.

The Use of the Matching Rule in a Non-numeric Proportional Reasoning Task

We first observed the use of the Matching Rule in a non-numeric proportional reasoning task with 7-th grade children. A brief description of some items of the task and two of the strategies used by the children to solve these items are necessary here for the discussion. The task involved two pairs of blocks (A, B) and (C, D) like those shown in Table 1, where blocks A and C were constructed from one kind of unit-blocks, blocks B and D from another. The unit-blocks used in A and C were larger in size than the unit-blocks in B and D. The number of unit-blocks in A was less than the number of unit-blocks in C, and remained constant across tasks. Three different instances of blocks B and D were used. The number of unit-blocks in B and D was one less, the same, or one more compared to the number of unit-blocks in A and C, respectively. The subjects were given a statement about the weight relationships between A and B (<, =, >), then were asked to judge the weight relationship between C and D.

If the Matching rule was used, the child would begin by looking at the relationship within pairs of blocks (A, B) and (C, D). The child would first notice that the number of unit blocks in the decks of these blocks is equal in number (see Table 1), then would determine the number relationship between blocks A and B, N(A, B), and the number relationship between blocks C and D,

N(C, D). In the next step the child would acknowledge the given weight relationships between blocks A and B, W(A, B), and attend to the question of the required weight relationship between blocks C and D, W(C, D). The child would then observe one of two relations between relationships: One was that the number relationship (<, =, or >) between A and B and between C and D are the same relation, i.e., N(A, B)=N(C, D), the other was that the number and weight relationships (<, =, or >) between A and B are the same relation, i.e., N(A, B)=W(A, B). Depending on which relationship was determined by a child, one of two rules was used. One rule is:

N(A, B)=N(C, D) implies W(C, D)=W(A, B).

As an example, suppose that block A has fewer unit-blocks than block B, block C has fewer unit-blocks than block D, and the weight of A is greater than the weight of B (see item 2 in Table 1); that is N(A, B) and N(C, D) would be the less than relation and W(A, B) the greater than relation. In applying this rule, the child would say something to the effect that, since N(A, B)=N(C, D), then W(C, D)=W(A, B), which means that the weight of C is greater than the weight of D. Notice that this rule is exactly the first formulation of the Matching Rule with $N=R_1$ and $W=R_2$.

Another rule which children applied to solve this problem is that

N(A, B)=W(A, B) implies W(C, D)=N(C, D),

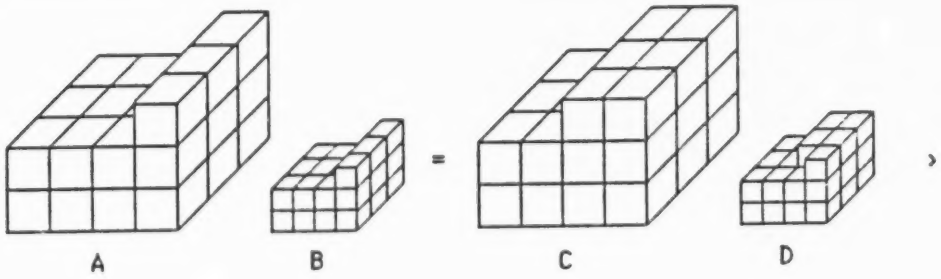
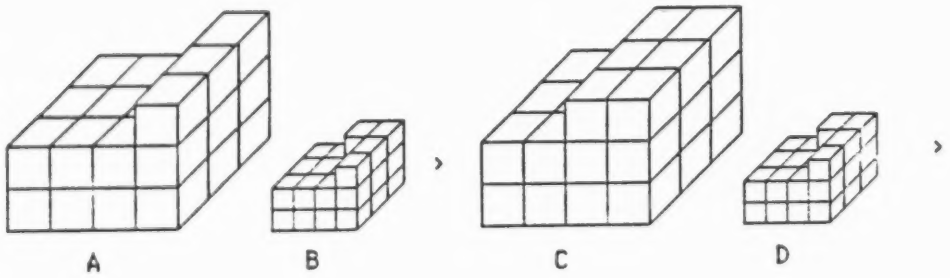
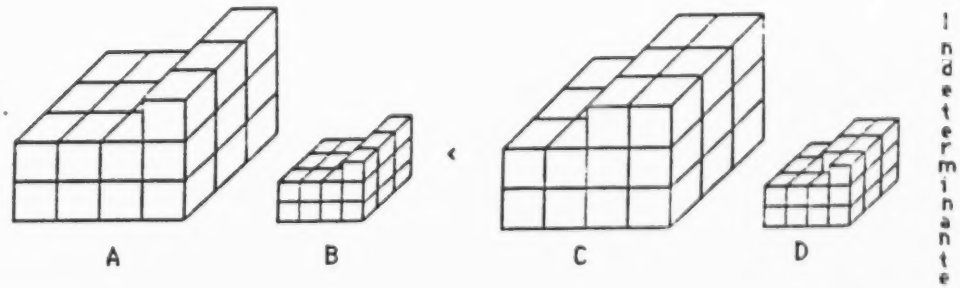
which fits the second formulation of the Matching Rule with $N=R_2$ and $W=R_1$. For example, if blocks A and B have the same number of unit-blocks, block C has more unit-blocks than block D, and blocks A and B weigh the same (see item 1 in Table 1), the child would say something to the effect that since N(A, B) and W(A, B) are the same relation, i.e., both are the equality relation, N(C, D) and W(C, D) must also be the same relation; that is, since N(C, D) is the greater than relation, W(C, D) will also be the greater than relation, thus C is heavier than D.

Some items (for instance item 3 in Table 1) cannot be solved by the Matching rule because neither one of the sufficient conditions N(A, B)=N(C, D) or N(A, B)=W(A, B) in the above two rules holds. This posed a problem to those children who depended on this strategy. After finding they were unable to solve a problem using the Matching strategy, one of two avenues were taken. Either the children would use a fall back strategy (i.e., fall back to a less sophisticated strategy) or would use a derivative of the Matching rule, which we call the Imposed Matching rule. When using this rule, the child would mentally substitute one of the given blocks for another block so that the condition N(A, B)=N(C, D) holds and therefore the first rule could be applied to conclude W(C, D)=W(A, B). Based on this conclusion and using transitive reasoning, the weight relationship

between the given blocks C and D could be determined.

Table 1

Description of the blocks tasks used.

Item	Pair (A, B) presented	Given weight relationship between A and B	Pair (C, D) presented	Correct weight relationship between C & D to be found
1.				
2.				
3.				

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Applications of the Matching Rule

Earlier we showed how the Matching Rule can explain the error children use to make in comparing two decimal numbers. In this section we will demonstrate the application of this rule in the domain of algebra, space does not permit inclusion of examples from other domains; they will be reported at another time. We start with the common error which results from generalizing distribution over addition, reported by Matz (1982). This error fits the Matching Rule in the following way. The student observes that two mathematical expressions structured like $2(x+y)$ and $2x+2y$ have the identity relation between them. That is, if R_2 is the order relation and $A=2(x+y)$ and $B=2x+2y$, then $R_2(A, B)$ is the identity relation. When the student encounters other mathematical expressions such as $(x+y)^2$ and x^2+y^2, he or she would observe that the structural relationship between the two expressions $2(x+y)$ and $2x+2y$ is the same as between $(x+y)^2$ and x^2+y^2. Denoting this structural relationship by R_1, we can encode this student's syntactical observation by $R_1(A, B)= R_1(C, D)$, where $C=(x+y)^2$ and $D=x^2+y^2$. Then, according to the Matching Rule $R_2(C, D)=R_2(A, B)$; that is, $R_2(C, D)$ must also be the identity relation, which means $(x+y)^2=x^2+y^2$. Application of the Matching Rule to conclude that $\sqrt{x+y}=\sqrt{x}+\sqrt{y}$ can be explained in the same way.

Another type of error identified by Matz is that students fail to realize the critical nature of the 0 in the equation $(x-a)(x-b)=0$ when they apply the solution:

$(x-a)=0$ or $(x-b)=0$

$x=a$ or $x=b$.

Consequently, they solve any equation of the form $(x-a)(x-b)=k$ by the same procedure, substituting 0 by K. This error can also be explained by the Matching Rule in the following way. This situation, from the student's point of view, involves the equation structure, $R_1(XY, Z)$="XY=Z," and the relation, $R_2(XY, Z)$="X=Z or Y=Z," where X and Y are both linear expressions and Z is a real number. The student perceives that $R_1(XY, 0)=R_1(XY, k)$ for any real number k, and he or she knows that the relation $R_2(XY, 0)$="X=0 or Y=0" holds. Consequently, using the Matching Rule the relation $R_2(XY, k)$="X=k or Y=k" must hold for any real number k.

The Imposed Matching Rule

The technique of rule extrapolation (Matz, 1982) can also be used to explain correct problem solving behavior. There are two ways to proceed, in solving a new problem. If the student has an applicable rule, an answer can be generated simply by completing the rules action. This is the case for the matching strategy in some of the blocks tasks. However, if rules currently known by the student do not apply, and modification of the rule to fit the situation does not seem feasible, then the student is faced with a more creative kind of problem solving -- using the extrapolation technique of modifying the situation to fit the rule. An example of this extrapolation technique is seen in the Imposed Matching Rule. Modification of the problem situation took place in this strategy when in some items of the Blocks Task (see item 3 in Table 1) the child observed that the necessary conditions of the Matching Rule are not satisfied. Some children used hypothetical reasoning to change the problem situation so that the matching strategy would apply and then modified the result after applying the rule, to take account of the supposition made to modify the problem situation. Because of space limitation, elaboration on the application of this strategy will appear in forthcoming paper.

Conclusions

It is important to recognize that cognitive conflict is not a negative aspect of learning. It is likely that no learning takes place unless some degree of conflict exists. Broadly speaking, cognitive conflict can be associated with a Piagetian construct of disequilibrium. Cognitive conflict, it would appear, arises whenever the learner is aware of discrepancy between the cognitive structure he or she possess and the cognitive structures (i.e., knowledge and understanding) he or she would like, or needs, to have. Cognitive conflict results in a state of disequilibrium, which in turn, results in motivation to learn, or if the cognitive conflict is excessive, it would lead to withdraw or anxiety. For educational practice the issue is on the questions of: (a) how to create situations which establish appropriate level of cognitive conflict for individual learners and (b) how to guide students to conflict resolutions through the construction of knowledge structures which are consistent with domain principles.

In order to answer these questions for educational practice, research in mathematical cognition needs continued emphasis on:

1. Gaining insight into children knowledge structure in varies mathematical domains.
2. Analysis of mathematical domain from both logical and cognitive prospectives.
3. Identification of cognitive structures which seem to be necessary for expertise in these

mathematical domains.

4. The development of experience, based on the knowledge of the above, which cause cognitive conflict and lead toward conflict resolution.

5. how to (a) inhibit the learners invention or construction of "buggy" strategies and (b) facilitate the learners invention or construction of domain consistent strategies.

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Inconsistencies in Preservice Elementary Teachers' Beliefs and about Multiplication and Division

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The authors' research concerning the difficulties preservice elementary teachers have in selecting the appropriate operation to solve multiplication or division word problems involving decimals led to the observation of a number of inconsistencies in the beliefs and skills possessed by some preservice elementary teachers. For purposes of this paper, these apparent inconsistencies are classified into three broad categories:

Inconsistencies between explicit and implicit beliefs about the operations,
Inconsistencies between beliefs and computational skills, and
Inconsistencies among explicit beliefs about the operations.

Probable sources of the inconsistencies are described as are suggestions for using the inconsistencies to help preservice elementary teachers achieve a cognitive conflict that can lead to concept reformation.

BACKGROUND

A growing number of investigators (e.g., Bell, Swan, & Taylor 1981; Fischbein, Deri, Nello, & Marino, 1985; Greer & Magan 1986; Owens 1987; Sowder, 1986) have pointed to the difficulties children, adolescents, and even adults have in solving word problems that involve multiplication and division with decimals. Researchers have documented the difficulties and theorized and probed their causes. The phenomena of "nonconservation of operation" has been identified (Greer, 1987), and teaching experiments have been conducted to help students overcome their misconceptions. The authors' research has pursued a similar path, albeit with a specific population, preservice elementary teachers.

Our research in the area of multiplication and division has addressed a number of questions including:

1) Do the primitive models of multiplication and division (as defined by Fischbein et al., 1985) influence preservice elementary teachers' performance in solving word problems?

2) Are preservice elementary teachers' beliefs about multiplication and division implicit or explicit?
3) What seems to support preservice teachers' beliefs, their reliance on the primitive models, and their related misconceptions?

Our early work (Graeber, Tirosh, & Glover, in press) indicated that preservice elementary teachers did indeed seem to be influenced by the primitive models of multiplication and division. Later work (Graeber, Tirosh, & Glover, 1986) suggested that the primitive partitive model of division appeared to dominate preservice teachers' thinking about division.

In the course of analyzing the data from these and subsequent investigations, the authors became more and more aware of apparent contradictions between the skills exhibited and beliefs professed, and even among beliefs held, by the same individual.

PROCEDURE AND METHODS

The data reported in this paper come from both written instruments and interviews. Written instruments were used to gather data on preservice elementary teachers' 1) reactions to generalizations about the operations and the reasons for their beliefs, 2) success in completing multiplication and division exercises (calculations) involving decimals, 3) performance in writing word problems for given division expressions, and 4) success in writing mathematical expressions for word problems. Over 85 of these preservice teachers were interviewed.

During a regular class meeting the preservice teachers first completed one instrument that covered the beliefs, calculations, and writing of word problems. In a subsequent class meeting they wrote expressions for approximately 16 word problems, 13 of which were multiplication or division problems. Individual, tape-recorded interviews were conducted within three to four weeks of the administration of the written instruments. Typically the subjects were given a number of word problems similar to those missed earlier and asked to tell what expression they would use to solve it. The interviewer spent 30-50 minutes probing the student's beliefs and understandings.

The data reported here come from 136 preservice elementary teachers most of whom were college

juniors, and all but one of whom were female. All 136 subjects completed the same written instruments probing beliefs, computation, and ability to write word problems. Since different forms of word problem samples, designed to gain information about specific misconceptions, were used, the number of subjects responding to any one set of word problems varied from 28-40. Results on some items common to a number of the forms and on later administrations of the same sets of word problems to other samples of preservice elementary teachers indicated that the relative difficulty of the different types of problems held across larger samples.

NOTED INCONSISTENCIES

In the following discussion, the preservice teachers reactions to statements about the operations (e.g., "In a division example, the quotient must be less than the dividend.") are assumed to reveal their explicit beliefs. In contrast, the preservice teachers' responses in writing expressions to solve word problems and the justifications they offered in describing their selection of an operation for a word problem are assumed to reveal their implicit beliefs.

Inconsistencies between explicit and implicit beliefs

There were substantial discrepancies in the percent of students who responded correctly to an explicit statement about multiplication or division and the percent who used that knowledge in word problem solving settings. For example, 85% of the preservice elementary teachers correctly indicated that the product in a multiplication problem is not always greater than either factor. Nevertheless, when asked to write an expression for a word problem involving a decimal operator less than one (e.g., One bolt of silk fabric costs $15,000. What is the price of .75 of the bolt?), 45% responded incorrectly. The vast majority of the incorrect responses were 15,000 ÷ .75. During interviews the subjects argued that they chose division for this example because .75 of a bolt would cost less than a whole bolt, less the $15,000. So one must divide.

Although 87% of the students agreed that the divisor can be larger than the dividend, somewhat more than 50% of the preservice elementary teachers

reversed the role of divisor and dividend when writing an expression to solve word problems such as "12 friends together bought 5 pounds of cookies, How many pounds did each one get if they each got the same amount?" When these students were asked why they wrote the expressions as they did, they almost invariably told us that they did so, to use one student's words, "because this kind of problem always goes like that, the little into the big."

Inconsistencies between beliefs and computational skills

Another inconsistency becomes apparent when one compares computational skill with responses to an explicit statement about division. Although 87% of the 136 correctly computed 5 as the quotient of $.75\overline{)3.75}$, only 45% of the same subjects disagreed with the statement, in a division problem, the quotient must be less than the dividend.

The parallel inconsistency for multiplication did not arise. About 90% of the students correctly completed a multiplication example with a factor less than one, and 85% correctly labeled as false the notion that the product must be greater than either factor.

With the exception of the inconsistency concerning the relative size of the quotient and the dividend, the students' success with the computational examples was generally similar to their success in responding to the related explicit statements of belief. Given the data discussed in the immediately preceeding section, this puts their computational skills at variance with a measure of their implicit beliefs, their performance in writing expressions for word problems. For example, the 50% of the preservice teachers who wrote the expression $12 \div 5$ for the friends and cookies problem described above seem to believe that the divisor must be smaller than the dividend. However 89% of the preservice teachers correctly computed a quotient for the example, $15\overline{)5}$.

Inconsistencies among explicit beliefs about the operations

Some may feel that the inconsistencies described above are easily explained by the fact that different types of knowledges are being compared. More vexing then, perhaps, are apparent

discrepancies within students' reactions to explicit statements about the operations.

If one accepts multiplication and division as inverse operations, the two statements "multiplication always makes bigger" and "division always makes smaller" are logically equivalent. The preservice teachers written justifications for explicit statements and their comments during interviews indicated that they did recognize multiplication and division as inverse operations. Further their comments during interviews and the expressions written for word problems suggested that the beliefs "multiplication always makes bigger" and "division always makes smaller" are closely tied--perhaps to the point of being one belief. All the subjects were asked to label as "True" or "False" the statements, "In a multiplication problem, the product is always greater than either factor." and "In a division problem, the quotient must be less than the dividend." Interestingly, 45% of the preservice teachers successfully refuted the stated misbelief about multiplication but did not refute the misbelief about division; only 4% correctly refuted the misbelief about division but did not refute the misbelief about multiplication. Thus, although it appears that the preservice elementary teachers understand these operations as inverses, almost 50% of them were inconsistent in their treatment of two statements. The situation is no less complicated if one looks at student responses to four of the statements about division. The four related statements appearing on the same instrument were:

a) In a division problem, the quotient must be less than the divisor.
b) In a division problem, the divisor must be a whole number.
c) The quotient for the problem 60/.65 is greater than 60.
d) The quotient for the problem $70 \div 1/2$ is less than 70.

Thirty-one percent of the preservice elementary teachers gave answers that were consistent across all four statements. Thirty-six percent agreed that the quotient must be less than the dividend but responded correctly to at least one of the statements about the quotient and the dividend in 60/.65 or $70 \div 1/2$. Sixteen percent agreed that the divisor must be a whole number but responded to at least one of the statements about 60/.65 or $70 \div 1/2$ correctly.

PROBABLE SOURCES OF INCONSISTENCIES

The major sources of these inconsistencies seem to be the preservice elementary teachers': (1) dependence on the domain of whole numbers and the primitive models of the operations and (2) reliance on procedural knowledge (Hiebert, 1986) that dominated, or at least was not linked to, correct conceptual knowledge (Tirosh & Graeber, 1987).

These sources were identified from the justifications students wrote and the reasoning they evidenced during interviews. In some cases procedural knowledge supports the primitive models. For example, one does not divide by a fraction, one inverts and multiplies. Similarly one does not divide by a decimal, one "changes" the decimal to a whole number. This later procedure conforms to the notion that the divisor must be a whole number and assures that the quotient is smaller than the transformed dividend. Thus, it is difficult to untangle the influence of these two sources. In any event it does seem clear that many preservice elementary teachers have not integrated their procedural and conceptual knowledge of these operations.

OVERCOMING MISBELIEFS

Many researchers in both science and mathematics education have noted the difficulty of changing students cognitive beliefs (Owens, 1987; Sowder, 1986). These observations are consistent with learning theorists' assessment of the difference between acquiring and restructuring knowledge. Acquisition of information consistent with the student's existing knowledge, appears easier than acquisition that requires conceptual change or reorganization. Teacher educators hope that preservice teachers who hold inconsistent beliefs and misbeliefs such as those described above will change their existing knowledge.

One method that has shown some promise in helping preservice teachers (Tirosh, Graeber, & Tirosh, 1987) and adolescents (Swan, 1983) overcome the influence of such misconceptions is the "conflict teaching" method described by Swan. Conflict is also a key element in what Vosniadou and Brewer (1987) called the "Socratic dialogue" method. These conflict approaches stem from Piagetian theory. Particularly helpful to us was Flavell's (1977)

notion that cognitive conflict results in growth if the learner (1) notices both of two conflicting elements simultaneously, (2) appreciates the inherent conflict in the two, (3) searches for a resolution, and (4) achieves a conceptualization that resolves the conflict. If one accepts the need for all four of the steps Flavell identifies, early stages of any instructional intervention depend on whether the misbeliefs are implicitly or explicitly held. For misbeliefs that are explicitly held, e.g.," the quotient must be less than the dividend," preservice elementary teachers computational skills may be used to design a situation in which their belief is challenged by their computation. For those beliefs that many preservice teachers hold only implicitly, e.g., "the divisor must be less than the dividend" or "the product is greater than either factor" the first task is to make the preservice elementary teachers aware of their behavior and the likely implicit misbelief.

Our experience in clinical interviews suggests that the intervention must indeed follow students beyond this stage of awareness. For example, a sticking point in some clinical interviews came when students recognized two conflicting elements, but searched for a means of eliminating the conflict -- e.g., changed their computation to accomodate their misbelief. In individual interviews, some preservice teachers who first computed $.5\overline{)4}$ as 8 and were then confronted with their statement that the quotient must be less than the divisor, decided that that the answer must be .8. These interviews also seemed to confirm the need for attending to Flavell's third and fourth elements. Without a firm hold on the measurement interpretation of division, little meaning can be brought to $.5\overline{)4}$. Lack of a meaningful interpretation seemingly reduced both the desire and likelihood of finding a means to resolve the apparent conflict.

We recognized two other sources of conflict that might be exploited to induce change. Since many preservice teachers apparently understand the inverse relationship of multiplication and division and can refute the explicit statement that "multiplication always makes bigger," there is a basis for conflict with the more popular notion that "division always makes smaller." Our serious reservations about this approach result from the fact that many preservice teachers hold the misbelief "multiplication makes bigger" implicitly. Another possible approach, as yet untried by the

authors, would be to actively involve preservice teachers in contrasting the operations in the domain of whole numbers with the operations in the domain of positive rationals.

Vosniadou and Brewer (1987) discuss "analogies, metaphors and physical models" as other means for restructuring knowledge. Although the analogy technique might suggest asking learners to select the operation to solve a whole number word problem and then to select the operation for a parallel problem with decimals less than one, the phenomena of nonconservation of number has caused a number of researchers to reject this route. Bell, Swan, & Taylor (1981) and the authors have been somewhat discouraged from using diagrams by students limited use of diagrams or inability to produce diagrams.

In any discussion of overcoming misbeliefs, some attention can be directed toward the possibility of preventing them. Given the apparent logic of beginning mathematics instruction in the domain of whole numbers and a constructionist point of view, misbeliefs such as "multiplication always makes bigger" may not be preventable. However, instruction in the middle school grades can seemingly provide for more opportunities than presently exist to apply the measurement interpretation to division by positive decimals less than one and to contrast the operations of multiplication and division in the domain of whole numbers with the same operations in the domain of rationals. Use of calculators and the move toward introducing the meaning of decimals at earlier grade levels suggest that the division of a smaller whole number by a larger whole number can be introduced earlier in the curriculum and used more frequently. The greatest opportunity for making possible control of misconceptions may be in providing instruction that gives conceptual meaning to the operations and links these meanings to the procedural rules for computation in the domains of whole and rational numbers.

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Inconsistencies in Preservice Elementary Teachers' Conceptions about Multiplication and Division: What, Why, and How?

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What inconsistent ideas about multiplication and division do preservice elementary teachers hold?

- 85% argued that in a multiplication problem the product is **not** always greater than either factor. Yet, 40-45% indicated that you find the cost of .75 of a bolt, by dividing the cost of the whole bolt by .75, because you want an answer less than the cost of the whole bolt.

- 87% argued that the divisor can be greater than the dividend and 89% correctly computed $15\overline{)5.0}$. However, about 50% indicated that you find the answer for "12 friends shared 5 pounds of cookies, how many pounds did each get?" with the expression $12 \div 5$.

- 87% correctly computed $.75\overline{)\ 3.75}$ as 5. But, 55% argued that the quotient must be smaller than the dividend.

- 45% argued that the product in multiplication is not always greater than either factor and yet agreed that the quotient must be less than the dividend.

- 36% agreed that the quotient must be less than the dividend and yet correctly identified the relative size of the quotient and dividend for $60/.65$ or $70 \div 1/2$.

- 16% argued that the divisor must be a whole number any yet correctly identified the relative size of the quotient and dividend for $60/.65$ or $70 \div 1/2$.

Why do the preservice teachers hold these beliefs?

Interviews and the preservice teachers' written justifications suggest that the inconsistencies are result of their reliance on the domain of whole numbers and the primitive models of the operations.

Further, these primitive models are frequently supported by a procedural understanding of the operations. Taught or constructed procedural rules such as, "You don't divide by a decimal you

move the decimal point and change it to a whole number," support the primitive models.

Attention to computational form also provides one possible explanation of inconsistencies among the preservice teachers beliefs. While only 45% of the students disagreed with the statement "In a division problem the quotient must be less than the dividend," 85% took exception to the statement In a multiplication problem, the product must be greater than either factor." This discrepancy exists despite the fact that the inverse relationship between the two operations appears to be accepted. Consider that in the standard written algorithm for multiplication by a positive decimal less than one, the decimal factor and final product appear intact at the end of the computation. However, in division by a decimal less than one, the algorithm alters the dividend so that the quotient always is less than the transformed dividend.

A second major hypothesis about the inconsistencies comes from the fact that students can execute computations that counter their stated beliefs about the operations. This suggests that their procedural knowledge (as defined in Hiebert, 1986) is not linked to correct conceptual knowledge. What Hiebert terms "compartmentalized knowledge" seems to be a major culprit.

How can awareness of inconsistencies be used to promote learning?

We chose to answer two questions: (a) Can the learner's awareness of inconsistencies be used to promote learning?, and (b) Can the educators' awareness of the existence of these inconsistencies prevent their occurrence or help learners become aware of the inconsistencies?

a) We believe, and have some evidence to support the notion, that a preservice teacher's awareness of inconsistent beliefs can be a first step in promoting learning. Swan's "conflict teaching method" appears to be one mechanism for enabling learners to realize inconsistencies.

However this awareness must be accompanied by a recognition that the disparate beliefs represent a "both cannot be true" situation. Further, the learners must desire and find a means of reconciling the inconsistencies. This means is not apt to be sought or to be able to be provided

unless the disparate components each have meanings that can be linked. The desire to overcome inconsistencies is also apt to be related to learners' more global beliefs about mathematics -- is mathematics supposed to be consistent?

b) Awareness of popular learner inconsistencies may not enable the best intentioned and most skilled educator to prevent learners from developing these inconsistencies. Certainly as long as the curriculum introduces the arithmetic operations within the domain of whole numbers, beliefs such as "the quotient is always less than the dividend" are apt to temporarily reside, at least implicitly, alongside conflicting computational knowledge, such as $4 \div 1/2 = 8$.

Educators must also consider the hypotheses of Posner, Strike, Hewson, & Gertzog (1982) that it is the learner's resistance to a conceptual change that could accomodate inconsistencies that leads to the very compartmentalization identified as a culprit. Or, is it more frequently the case that learners are not disturbed by, and do not even implicitly sense, the inconsistencies? Do they simply store different conceptual frames?

In any event, we contend that unless preservice teachers become aware of their own misconceptions and inconsistencies, some inconsistencies may go unchallenged from teacher to student. Inconsistencies are not easily diagnosed unless evaluations are specifically designed to detect them. Only if the likelihood of such inconsistencies and the threat they pose for students' learning and performance is well understood by teachers, are teachers apt to carry out a diagnosis.

Further we suggest that there are many steps that can be taken by curriculum writers and teachers to lessen the grip that the primitive models and emphasis on procedural understanding have in supporting (or creating) the inconsistencies discussed. First, teachers and materials can provide meaningful models and interpretations for the operations that can be developed in the domain of whole numbers and then transferred to the domain of rationals (e.g., an area model for multiplication, the measurement interpretation of division). Second the emphasis on procedural understanding of the operations, should be accompanied by an equally strong conceptual understanding of the operations and the linkages must be made explicit.

The Relationship of Students Definitions and Example Choices in Geoemetry

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Mathematicians and mathematics teachers rely heavily on definitions, examples and nonexamples in order to communicate ideas and to advance knowledge. While definitions may vary and can be challenged (Lakatos, 1976), they provide a description of a mathematical concept. Examples illustrate the definition and provide special cases. Nonexamples do not satisfy the conditions of the definition and often provide a helpful contrast. Together, definitions, examples and nonexamples enable a mathematician or a mathematics student to generalize ideas and to transfer information to new contexts.

Many mathematics learners are not able to utilize the power of a definition or its relationship to examples. Classroom teachers are well aware that a student may be able to state an acceptable definition of a mathematical concept and still be unable to choose examples correctly. The examples students' choose or the ideas they build may be inconsistent with their own definitions. While there is evidence that inconsistencies exist, we do not know how students perceive these inconsistencies or if they are even aware that they exist.

Middle school students' geometric concepts, definitions and examples were investigated. There was evidence of inconsistencies from the researcher's perspective but many of the ideas did not appear inconsistent to the student. The relationships of student generated definitions, examples and concepts were explored.

We need to know more about how students create and use their own definitions. We need to know more about students' inconsistent ideas and how they influence students' mathematical constructions. A related issue is what stimulates cognitive conflict and when is such conflict helpful.

Research in both psychology and mathematics education offers a foundation. Some researchers (Piaget & Inhelder, 1964; Winer, G. 1980) claim that a child's inability to recognize inconsistencies is related to cognitive development and learning is stimulated when students reach a state of disequilibrium or cognitive conflict. The van Hieles (1984) have identified levels of geometric thought that a student must achieve in a given sequence. The levels of thought are based on student awareness of geometric properties and relationships, and their abilities to operate with them. Another approach suggests learners form prototypes that guide their construction of concepts (Kellogg 1980; Anderson, 1980) and these prototypes are filled with irrelevant and sometimes erroneous information (Burger, 1986; Burger & Shaughnessy, 1986) which may lead to inconsistent ideas. Formal definitions may have little or no meaning for the

student's mathematical structure. Previous work with elementary school students and geometry concepts (Wilson, 1986b) has suggested that not only were students' examples inconsistent with conventional mathematics definitions, but students' examples were inconsistent with their own definitions. For instance, seven of the 49 elementary students defined a triangle rather than a rectangle but chose examples of rectangles rather than being consistent with their own definitions. Most students did not view squares as special cases of rectangles even when their definitions of rectangle allowed for such an interpretation. In contrast, some students defined rectangles as "long squares" but chose squares as examples of rectangles.

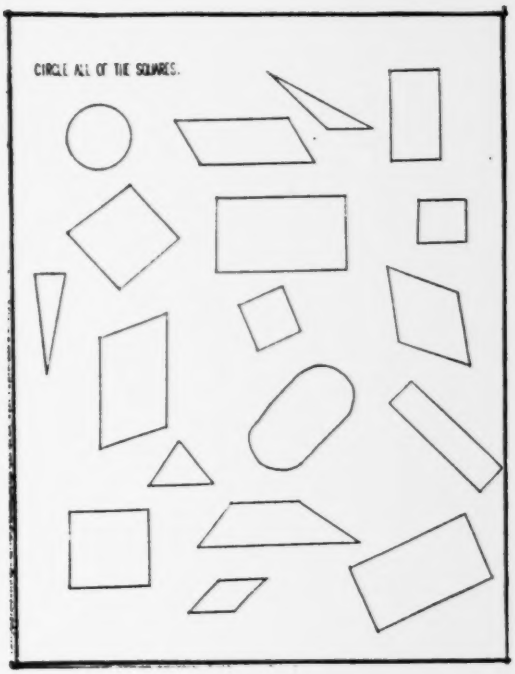
Although middle school students have a broader mathematical vocabulary than elementary students and most have developed to a semi-concrete stage of operation, sixth and eighth grade students exhibited some of the same inconsistencies found in elementary children. By the time students reach middle school, they have structured several geometric concepts at an informal level and have had some formal treatment of rectangles. They have constructed prototypes that contain both relevant and irrelevant features. Interviews suggested reasons for some of the inconsistencies.

Methods and Procedures

Students in three 6th grade classes and three 8th grade classes from each of the schools that participated in the study. Two methods of data collection were used.

Paper and pencil instrument. A paper and pencil test was administered by each classroom teacher. The test had four components. First, student drew three different triangles, three different rectangles and three different squares. In the second section, the students wrote a definition of the geometric terms "triangle", "rectangle", and "square" in their own words. The third task involved collections of geometric figures. Students identified rectangles, squares and triangles by circling examples. In the fourth section, students are asked to agree or disagree with statements pertaining to properties of figures and relationships between figures (i.e., All squares are rectangles). Students were permitted to move freely between sections of the test.

Follow-up interviews. Two students from each class were selected to be interviewed based on the types of errors on the test. Since not all students were present on interview day and some permission forms were not returned, only 21 of the 24 students were interviewed.



READ EACH SENTENCE. DO YOU AGREE, DISAGREE OR ARE NOT SURE?
PUT AN "X" NEXT TO ONE CHOICE.

1. ALL RECTANGLES ARE SQUARES.
   AGREE _____
   DISAGREE _____
   NOT SURE _____
2. ALL SQUARES ARE RECTANGLES.
   AGREE _____
   DISAGREE _____
   NOT SURE _____
3. ALL CLOSED FIGURES THAT HAVE 4 SIDES AND 4 RIGHT ANGLES (4 SQUARE CORNERS) ARE RECTANGLES.
   AGREE _____
   DISAGREE _____
   NOT SURE _____
4. ALL FIGURES THAT HAVE 4 SIDES WITH THE SAME LENGTH ARE SQUARES.
   AGREE _____
   DISAGREE _____
   NOT SURE _____
5. ALL TRIANGLES HAVE THREE SIDES.
   AGREE _____
   DISAGREE _____
   NOT SURE _____

Figure 1

The interviews lasted approximately 40 minutes and focused on the relationship between definitions and examples. The interviews were video and audio taped. All interviews were conducted by the researcher. The interview began along the lines of a given script but often deviated to explore a particular question. The students were questioned about what they thought and how they came to their conclusions. Inconsistent ideas were probed in a way that allowed the students to explain what they were thinking.

Data Analysis

All responses on the paper and pencil test were coded and recorded in a data file that could be used to count specific responses or combinations of responses. The drawing were rated on three scales: mathematical accuracy, degree of variety in size or shape, and degree of variety in orientation. Counts were made of specific properties required by student definitions (i.e., parallel sides, right angles, long).

After coding the data, a team of three researchers discussed possible inconsistencies. We noted areas where students seemed to be inconsistent in their responses.

The tapes were viewed looking for insight on inconsistencies found in the paper and pencil test. The tapes also produced new inconsistencies that were not evident in the test. Frequent team meetings were held to discuss the prevalency of a specific inconsistent pattern and we noted differences between students and between grade levels. Often we generated new questions and

returned to the tapes looking for more insight. Numerous theories were generated and we returned to the paper and pencil test for support or reformulation. We were careful to note when inconsistencies were common across students, but we did not eliminate any individual inconsistent ideas.

Findings

Students do generate inconsistent ideas as they construct mathematics. Sometimes they are aware of the inconsistencies and sometimes they are not. When they are unaware of the inconsistency, it may be because:

1) They do not have an inconsistency in their scheme. The language and logic that they use makes the ideas consistent. The inconsistency is from the researchers viewpoint of the student's thinking. (This is quite different from being inconsistent with the researcher's mathematics.) For example, a student may be comfortable with the idea that all squares are rectangles and all rectangles are squares, but they are not equivalent classes of figures. The words "all", and "are" may have a very temporal meaning. Students made the logic work with words like "sometimes" and "regular".

2) The student may not have considered the problem or has not seen a situation from a perspective that highlights the inconsistency. For example, a student may not have ever considered that a rhombus and a square are both parallelograms with four sides of equal length.

When students are aware of inconsistent ideas, they are interested in resolving the inconsistency to some extent. The solution may not be to create a more consistent mathematics; the student may construct a logic that minimizes the inconsistency.

Alice originally disagreed with "All squares are rectangles." After some discussion about rectangles and squares she reconsidered.

Alice: I think squares are certain kinds of rectangles.
Interviewer: Why would squares be certain kinds of rectangles.
Alice: They are made like rectangles but they're small. All their sides are the same. Rectangles only two sides are the same...they (squares) are a certain kind of rectangle that are small.
Interviewer: So some squares could be rectangles?
Alice: Yes, all squares could be rectangles.
Interviewer: ... now if you went back to this page where you chose rectangles, would you choose any different ones this time? (She did not choose squares as rectangles.)

Alice: I don't think so. I think all rectangles are long and you usually think of a square as small.
Interviewer: So even though you think that squares could be rectangles, you wouldn't choose this one (pointing to a square) as a rectangle.
Alice: No.
Interviewer: Are there just some squares that are rectangles and others aren't?
Alice: They are just a certain kind of rectangles, that are shaped different and small.
Interviewer: So you won't choose these as rectangles but you are willing to put back here that squares are rectangles (she nods yes). So you are going to agree there but you aren't going to choose them here, so I'm a little confused.
Alice: I just circled the longer ones because most people think of the longer ones.
Interviewer: If we did it over again would you still just circle the long ones?
Alice: Yes.

Interviewer: A boy is a person but not all persons are boys.
Alice: Girls are persons but not all people are girls... It can go one way but not the other.
Interviewer: Could that be how squares and rectangles fit together?
Alice: I guess so, all squares are rectangles but not all rectangles are squares.
Interviewer: Now if we thought of it that way when we went to choose rectangles, which ones would you choose?
Alice: The long ones because they're not squares.

Alice's inconsistent statements were typical of several conversations. She seemed to be aware of some inconsistencies but her logic and knowledge allowed her to feel comfortable with her concepts.

Considering all twenty one interviews, a variety of sources of inconsistent ideas were identified.

1. Students do not think of definitions as necessary and sufficient conditions.

2. Students' vocabularies are not sufficient to express their ideas. Often their words are imprecise. There is some evidence that this is not merely neglect but "fluid words" can serve a necessary function. Ambiguity can be useful.

3. Students possess strong prototypes that must be incorporated into their construction of concepts. The prototypes seem to become more flexible with mathematical development.

4. Students' logic is not formal or deductive. They do not seem to have a way to test the validity of an argument other than if it fits for a particular situation at a particular time.

Considerations

The inconsistency of ideas is determined by the perspective of the viewer. What the teacher sees as inconsistent, may not be inconsistent to the student. If learners (teachers and students) consider each other's perspective of the inconsistent idea, each can benefit in their own thinking. For example, if a student is confronted with the inconsistency in defining a square as a closed figure with 4 equal sides and the fact that a figure with four equal sides may not be a square, the student tries to resolve the inconsistency. The teacher can offer tools such as vocabulary (rhombus, parallelogram, right angles), classification concepts (i.e., hierarchy, subset), and analogies (i.e., boys, girls, and people), but the learner must resolve the inconsistency in a way that makes sense to him or her. The restructuring takes time. The teacher needs to try to learn about the student's mathematical structures, so that he or she can offer challenges that test the consistency of the student's structure.

The findings do not translate immediately to helpful suggestions for classroom instruction. Open ended questions, activity oriented instruction, chances for free exploration, and frequent class discussions about mathematics offer opportunities to investigate children's constructions. At this point, we can say that students and teachers hold inconsistent ideas. These ideas are based on properties, relationships of properties, prototypes, and logical reasoning. It may be possible to resolve inconsistent ideas but it requires a complex restructuring of the concept over time.

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Paper and pencil tasks indicated apparent inconsistencies between student's concepts in different contexts such as drawing, defining, choosing figures or answering questions. While some inconsistencies were found, interviews gave insight into the reasoning students were using. Some actions that appeared to be inconsistent, began to look consistent from the students' point of view.

INCONSISTENT GEOMETRIC IDEAS IDENTIFIED IN RESEARCH

Inconsistent ideas exist within the learner's structure of mathematics. The ideas may be inconsistent between contexts or between points in time; the learner may be aware of the inconsistencies or may not. It is important to note that this paper focuses on inconsistencies in the student's structure of mathematics rather than inconsistencies between the student's mathematics and more conventional ideas. For example several students claimed that a rectangle must be long and that it can not have equal sides. Although this idea is inconsistent with conventional definitions of a rectangle, it would not be considered inconsistent within the student's structure of mathematics unless it conflicts with another student-held idea. The following examples illustrate inconsistencies within the students' structures.

Students have difficulty with the relationship of squares and rectangles.

Student ideas about squares and rectangles varied with the context in which they were working. Many student definitions did not allow squares to be rectangles because they required rectangles to be long, but for others their definitions permitted squares and yet they failed to choose squares as examples of rectangles.

Context	6th grade	8th grade
	n=143	n=145
Definitions permitted squares	53	103
Chose at least 1 square as an example of a rectangle	11	38
Drew a square as an example	7	15

In a written context, 92 students agreed that all squares are rectangles and yet almost half of those students did not circle squares as examples of rectangles.

Students were attentive to angles in some contexts but not others.

Angles were notably absent from definitions, but some students seemed to pay attention to angles when they chose examples.

Context	6th grade n=143	8th grade n=145
Definitions of square permitted Rhombi	124	97
Chose rhombi as squares	66	37
Agreed to the statement #4: "All figures that have 4 sides with the same length are squares."	113	107
Agreed to statement #4 AND did NOT choose nonsquare rhombi when choosing squares.	48	67

Students had similar inconsistencies when considering parallelograms and rectangles. In the sixth grade, 122 students did not require rectangles to have right angles in their definitions. In a consistent manner, 101 sixth graders chose parallelograms without right angles as examples of rectangles. However, in the eighth grade, 106 students still did not require rectangles to have right angles, but only 61 chose parallelograms without right angles as examples of rectangles. It seemed as though they were guided by an unwritten definition that required right angles in rectangles. Their prototype of rectangle did not include parallelograms that lacked right angles.

SOURCES OF INCONSISTENCIES

Interviews with 21 of the students helped to clarify what some students' were thinking. In some cases the students were aware of the inconsistency and in other cases the situations were not considered inconsistent. Often as students talked, they became aware of the inconsistency in their ideas and wanted to make changes. There were also situations where the students want to leave things just they way they were! From a researchers point of view, several sources of inconsistencies were present.

Students ideas about definitions

From the paper and pencil test it appeared that students did not use their own definitions to perform tasks. Their definitions seemed to be unrelated to choosing examples or

answering questions. However the interviews confirmed that students did think about the concepts across contexts. In particular several mentioned that they used their definitions to choose examples or help with other parts of the test. However students have nonconventional ideas about definitions.

Students do not understand necessary and sufficient conditions. One student claimed that it was not necessary to put right angles into the definition because everyone knows that a rectangle has right angles. He only chose figures with right angles, but he did not want to add it to the definition because it seemed unnecessary to him. Another student offered that writing definitions was like what you do in Spelling. "Sometimes ... I won't put the whole thing...cause I don't want to write anymore. And just try to make it shorter if you can, and don't make it so long." A sixth grade student struggled during the interview to differentiate squares from rectangles. At one point the student offered "squares are rectangles with 4 equal sides" and seemed quite pleased with the discovery. Later in the same interview, the concept was lost and he could not reconstruct the idea.

Students' words were imprecise and could mean several different things.

The ambiguity of words was useful to students since they could explain inconsistent ideas by changing the meaning of the word. For example, an 8th grader defined a rectangle as a "close figure where not all sides are the same length." The student clearly wanted the opposite sides the same length and adjacent sides different lengths. That was exactly what the statement meant to the author of that definition. Several students used the term corner; it could mean right angle in some situations and any angle in other situations. Consider the versatility of some of the following definitions for rectangle:

-- a longer shape with 4 long straight lines
-- long, 2 even and 2 even
-- 4 sided figure with all sides the same--sometimes
-- two long lines across ways connected with two shorter ones going up and down with 4 points.

One student claimed that "if the opposite sides of a rectangle face right opposite then they form right angles."

Students possess strong prototypes of figures

Students worked from a limited, inflexible prototype. The prototype become more flexible as the student gained understanding of the concept. Sometimes, the prototype helped the student to choose examples that were consistent with a conventional definition but were inconsistent with their own definition. For example, some students chose only examples of parallelograms with right angles even though their own definition did not require right angles. However prototypes also caused

confusion. Most students had an image of a long rectangle, and even though they could explain that a square has all the properties of a rectangle, they did not think of a square as a rectangle. One student stated, "Squares look different from rectangles but in words they are the same."

Many students expressed a preference for pictures. They enjoyed the drawing and circling examples sections of the test. They said they learned more by seeing pictures. Interestingly, they said they would teach using words even though they enjoyed pictures. Some said they would use both to teach.

Students had difficulty with logical reasoning.

Students had difficulty following or creating a deductive argument. This may have been a result of students being at low van Hiele levels with respect to rectangles. They could not always identify properties and struggled with the relationships of properties of rectangles. Students had difficulty expressing the hierarchy with squares, rectangles, and parallelograms. Many did not want to alter definitions or choices even though they themselves had explained the inconsistency. One student explained that his definition could include right angles because he was using right angles to choose rectangles, but declined when asked if he wanted to add it to the written definition.

In contrast student reasoning was more consistent with regard to familiar concepts. They could easily define, choose and talk about triangles. They seemed to have the necessary vocabulary. They understood the hierarchy between boys, girls, and people, but they could not transfer that to the relationship of squares and rectangles. It seemed that the problems of a restricted vocabulary and lack of experience with a variety of parallelograms interfered with their reasoning.

There were instances where the reasoning was very clever. One student explained extremely clear that all squares were rectangles and all rectangles were squares. He illustrated that any square could be stretched out so that it was long like a rectangle and any rectangle could be stretched up so that all sides were equal like a square. He had never considered the figures rigid!

LEARNING THROUGH INCONSISTENCIES

Can inconsistencies promote learning? Yes. Surprisingly, the learning may not be what the teacher expects. More than half of the interviews became involved in exploring the relationship of squares and rectangles. Students seemed aware of inconsistent ideas and yet were resistant to restructuring their ideas. If their inconsistencies were due to limited definitions, reasoning, prototypes, and vocabulary, their structure can not change quickly. None of the students left the interview with a

radically different view of the relationship of squares and rectangles. However, they seemed eager to pick up new words that helped them express ideas they had already formed. For example, "opposite sides parallel" was a helpful phrase that meant something to them. They were aware of more figures to consider and sort. Many had not thought about rhombi. As they entertained new ideas they began to alter their prototypes and used words like "regular rectangles" and "sometimes" squares are rectangles. This seemed to show a mode of restructuring ideas.

There is value in working in different contexts. While it appeared that students worked quite differently when they were defining, drawing or choosing examples, many of those interviewed were concerned about consistency across contexts. By varying the context during instruction and evaluation, the teacher will become more aware of students' mathematical structures.

Inconsistencies in the Learning of Calculus and Analysis

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1.Research Background

In the mid-seventies, as a lecturer in mathematics rather than a researcher in mathematics education, I first became involved in the difficulties of students learning the calculus and analysis. At that time my belief, which I suspect remains common amongst professional mathematicians, was that the difficulties could be eased by preparing materials in a logical and coherent way for students to understand. Exploratory investigations into students' conceptions revealed the inadequacy of this viewpoint as the inherent cognitive conflict in many of the concepts was exposed.

Inconsistencies identified in research

In Tall (1977) the results of the an investigation into student's beliefs were reported, based on written responses to questionnaires by a population of 36 mathematics students in their first week of study at university.

One item made the request:

> If you know the definition of the limit of a sequence, write it down: $s_n \to s$ as $n \to \infty$ means:

A later one asked:

> Is $0\cdot\dot{9}$ (nought point nine recurring) equal to one, or is it just less than one? Explain the reason behind your answer.

Only 10 out of 36 claimed to know a precise definition and only seven were able to formulate a definition that was mathematically acceptable. Of these seven, only *one* responded that $0\cdot\dot{9}=1$.

Thirteen of the thirty six held apparently conflicting views, asserting that $0\cdot\dot{9}$ was less than 1, whilst elsewhere stating that

$$\lim_{n\to\infty} (1+\tfrac{9}{10}+\tfrac{9}{10^2}+\ldots+\tfrac{9}{10^n}) = 2.$$

A week later the students were asked to write down various decimals as fractions, including

$0\cdot25$
$0\cdot05$
$0\cdot3$
$0\cdot333\ldots$
$0\cdot\dot{9} = 0\cdot999\ldots$

Two thirds of the students (24) now said that $0\cdot\dot{9} = 1$ (or 1/1), including 13 who had previously affirmed the result was less. Their written answers also exhibited the conflict in terms of crossings out and added comments.

Subsequent research by one of my Ph.D. students, Monaghan (1986), working with 16/17 year-olds studying the calculus, showed that "recurring decimals are perceived as dynamic, not static, entities and are not *proper* numbers. Similar attitudes exist towards infinitesimals when they are seen to exist". Comparing students taking a calculus course with students of a similar ability who were not, he concluded that "the first year of a calculus course has a negligible effect on students' conceptions of limits, infinity and real numbers".

Other investigations reveal a wide range of difficulties with the meaning of concepts in calculus and analysis (for example, Schwarzenberger & Tall 1978, Tall 1979a,b, 1980 a,b).

Cognitive conflict associated with limits of sequences, limits of functions and continuity is considered in Tall & Vinner (1981). The phenomena are here interpreted in terms of the theory of *concept image* and *concept definition*, defined as follows:

> We shall use the term *concept image* to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. ... As the concept image develops it need not be coherent at all times. ... We shall the portion of the concept image which is activated at a particular time the *evoked concept image*. At different times, seemingly conflicting images may be evoked. Only when conflicting aspects are evoked *simultaneously* need there be any actual sense of conflict or confusion.

On the other hand:

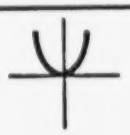
> The *concept definition* [is] a form of words used to specify that concept.
> (Tall & Vinner 1981, page 152)

The paper considers the curriculum studied earlier by pupils in an attempt to postulate reasons for the mismatch between students' evoked concept images and their knowledge of concept definitions. For example, the students had a concept image of a continuous function, which could have come from a variety of sources, not least being the colloquial meaning of the term in phrases such as "it rained continuously all day" (meaning there was no break in the rainfall). A questionnaire administered to 41 first year university mathematics students included the question:

Which of the following functions are continuous?

If possible, give reason for your answer.

$f_1(x) = x^2$



$f_2(x) = 1/x \quad (x \neq 0)$

$f_3(x) = \begin{cases} 0 & (x \leq 0) \\ x & (x \geq 0) \end{cases}$

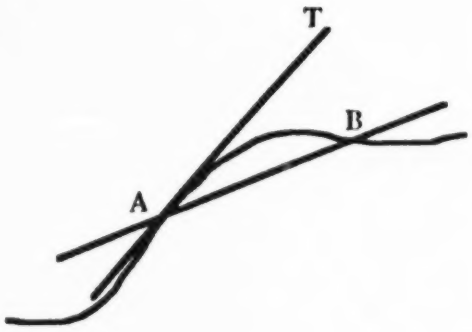
$f_4(x) = \begin{cases} 0 & (x \leq 0) \\ 1 & (x > 0) \end{cases}$

$f_5(x) = \begin{cases} 0 & \text{(rational)} \\ 1 & \text{(irrational)} \end{cases}$

Mathematically f_1, f_2 and f_3 are continuous, whilst f_4 and f_5 are not. But conceptual imagery suggests strongly that f_2 is *not* continuous (because its graph has a break) and other evoked images have a coercive effect on student responses. For example, f_3 is considered by some to be "not continuous" because "it is not given by a single formula", or because "it does not change smoothly at the origin".

In questionnaires it is quite possible for students to give "right answers for wrong reasons", or even "wrong answers for right reasons". Tall (1986a) asked students to respond in writing whether they thought the following statement to be true or false:

As B→A the line through AB tends to the tangent AT.
(True/False?)



Of a sample of nine 16 year old students interviewed in depth (as part of a larger project), four said the statement was "true" but linked the symbol B→A to vector notation and visualized B as moving to A *along the line BA*, so that the line (segment) BA "tends" to the tangent. Meanwhile, one student considered the statement "false" for the sensible reason that the lines were infinite so, way off at infinity, the line AB and tangent AT were still a long way apart no matter how close A and B became. For this reason questionnaires alone, without follow-up interviews, may not reveal the full story.

Vinner (1983) observed that students develop individual concept images of a tangent which conflict with the formal definition. In particular, early exposure limited to tangents to circles may cause them to see the tangent as a line which "touches the graph but does not cross it". Tall (1986a, 1987) used the computer to encourage discussion and a more visual approach to the tangent as part of a wider study in the calculus with a (pre- and post-test) comparison between 41 students aged 16/17 using the computer, 65 comparable students following a more traditional approach and 47 university students aged 18 (post-test only).

The computer helped provoke new insight into the notion of the tangent, though new conflicts could arise if not handled carefully (for instance, more students using the computer were now likely to believe that a tangent was a "line through two very close points on the graph" (because the program approximated the tangent in this way)).

The main thrust of the research in Tall (1986a) was to test a "cognitive approach to the calculus" using software that enabled the teacher to demonstrate, and the students to explore a wide range of examples of the gradient of a curved graph. There was significant improvement in the experimental students' ability to sketch gradients of given graphs, and their conceptualizations transferred to the more general case of the gradient and tangent of a graph at a point where the function was given by different formulae on either side.

2. Position Paper

On considering the evidence, it becomes very clear that the concepts under consideration are highly complex and not necessarily coherently understood even by teachers or professional mathematicians. Several papers (e.g. Tall 1980c, 1981a, 1981b, 1985b) have been devoted to the mathematical and cognitive problems with limits and infinity, including the existence of mathematical subcultures with conflicting views (such as standard and non-standard analysis, one denying the existence of infinitesimals, the other affirming it). The purpose of these papers was to underline the fact that there may not always be one universal mathematical truth which acts as a touchstone by which all others are judged. Furthermore, Monaghan (1986) demonstrated that these formal cultures are different again from students' belief structures, for instance, he shows that students' conceptions of the continuum "do not conform to the classical or the non-standard paradigms".

It was therefore in the early eighties that my attention turned to using the computer to provide a rich environment in which concepts could be demonstrated, explored and discussed. My thesis is that an environment allowing the user to explore both examples and non-examples of a mathematical concept or process can help the user abstract the general properties embodied in the examples and contrasted by the non-examples. An environment designed with this in mind is called a *generic organizer*, and (Tall 1986b) consists of a collection of generic organisers for visualizing calculus concepts. A new approach to the calculus using these organisers is described in a series of six articles in Mathematics Teaching, starting with Tall (1985a).

Computer software can provide a representation or model of the mathematical phenomena, but not always an exact translation. (For instance, pictures are drawn using finite pixels, so that straight lines do not normally look straight on a computer screen.) However, these obvious inconsistencies may be regarded as an *advantage*, not a hinderance. If the student can clearly *see* that the representation is not exact, it is possible to discuss the reasons why, and to begin to build up richer mental models.

It is my belief that we do students a disservice by organising the curriculum so that they are presented only with simple ideas first and given too great an exposure to an environment which contains regularities that do not hold in general. This just sows the seeds for later cognitive conflict. For example, doing geometry of curves only with circles, can give a dangerously limited idea of a tangent, or studying differentiation initially only with polynomials for so long may cause students to abstract general "rules" which are not true in a wider context. (For instance that at a maximum the derivative is zero, rather than *if* the derivative exists, *then* it is zero...)

There are two possibilities (which are not mutually exclusive): one is to research the cognitive conflict to be prepared to face it when it occurs, a second is to give a richer conceptualization from the start within which the conflict may later be framed. My approach is to design generic organisers to allow students to explore general ideas through specific examples and non-examples. In this way powerful general ideas can be introduced at the outset (such as differentiable and non-differentiable functions at the beginning of the calculus) and specific examples with pertinent

properties can be investigated (such as the tangent to a straight line or at an inflection point) to help students avoid narrow over-generalization.

This will need a radical reform of the curriculum. It applies at all levels of development. Much research, based on pre-computer environments, may be in error because it occurred in a context which may not pertain in future. For example, we teach young children about simple fractions in terms of halving and quartering because this is within their physical and mental capacity. But just because they may be *physically* incapable of dividing a cake into seven equal pieces does not mean that they are *mentally* incapable of visualizing it aided by appropriate software. A rich computer environment allowing children to carry out their mental ideas may give the opportunity to circumvent some of the trivializing introductions that may stunt future growth. We cannot make the complicated concepts more simple, but we can give far richer experiences that enable them to be seen in a wider, and more powerful, context. Recent research shows this to be a promising direction to follow (Tall & Thomas 1988).

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Cognitive Conflict in Learning Mathematics
Discussion of Papers

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Reading the papers was a pleasant experience because the author's have obviously taken the mathematical knowledge of students' seriously as well as how mathematical knowledge might be learned. I start the discussion with the proposition that mathematical learning consists in the modifications students make in their mathematical concepts. Mathematical concepts necessarily involve what David Tall calls a **concept image**. My view of a concept is a very liberal one and I give credit to Ernst von Glasersfeld for its formulation: A concept is any conceptual structure that can be re-presented in the absence of perceptual signals and is used in categorizing or relating experiential phenomena. A mathematical concept as I have defined it is compatible if not identical with what David called a concept image. His study of reasons for a mismatch between students' evoked concept image and their knowledge of concept definitions is at the heart of Pat Wilson's contribution as well, so I will start by commenting on those two papers.

I wonder what these sensitive researchers "see" in their student's mathematical language and actions--what the bases for their interpretations of their student's language and actions are and what concepts they attribute to their students? For example, David gives a wonderful example of students believing that $f(x) = 1/x$ where $x = 0$ is discontinuous because the graph of the function increases without bound as $x \to 0$ from the right and decreases without bound as $x \to 0$ from the left. This example of discontinuity is so strong that it still seems artificial to me to throw zero out of the domain of the function to "remove" the discontinuity. There has to be a time when a student can come to appreciate such powerful control over their mathematical experiences, but it seems to me to involve at least one level of abstraction above construction of the concept of continuous and discontinuous functions. My point is that we as researchers have only begun to decenter and to specify the itinerary of the abstractions that are involved in the construction of mathematical concepts by students and how their concepts might change from one level of abstraction to another. To ask a student to analyze their concept image using methods at a level above where they currently are can only leave them blocked without operations to perform because one is asking them to "step back" and observe themselves operate and to analyze the results of their operations.

There is no substitute for experiencing the inverse relation between x and f(x) as x becomes very small from the left of zero by actually making calculations and, on that basis, experiencing f(x) "plunging" downward without bound. Moving x inward toward zero by a specified method like choosing x to be the point 1/2 of the remaining distance between the current point and zero is indispensable in abstracting the powerful result that as x becomes very small in magnitude on the negative side of zero, f(x) becomes very large in magnitude in the negative direction. It is an even more astounding abstraction that what is going on is that x is actually **increasing** as it approaches 0 from the left and f(x) is actually **decreasing** without bound! This is, of course, the result of correlating magnitude and direction but it is crucial in forming the involved mathematical concept. Repeating the experience from the right of zero is just as rewarding.

The reason for discussing the particular example in detail is to emphasize that a concept image of a particular function should be thought of as the records or traces of the actor's conceptual operations used in construction. A concept image of a function is not a picture or a "graph", but "a program or recipe whose enactment constitutes the [concept image]--much in the same sense as Bridgman intended his "operational definitions" (von

Glasersfeld & Richards, 1983). I believe this is a liberating view of a mathematical concept because it provides us with point of departure in investigating mathematical knowledge of others--the first item of concern is to specify the involved conceptual operations in as much detail as possible. One cannot do this without observing the other in an environment and this is why I find David's idea of generic organizers through specific examples so powerful. **Our observations are dependent on our own ways and means of observation, and acknowledging that at the very beginning of our studies will lead us to the position of developing models of mathematical learning that are relative to the actions of the observer.** So, we just as well start off by creating possible learning environments that we believe favor active cognitive construction by the involved human beings because to do less does not somehow make our studies more "objective".

In Pat's paper, we can see the "figurative nature" of students' knowledge of rectangles and squares and its results. By and large, her subjects did not perform operations on their visualized images of what goes by those names. They seemed capable of performing such operations because one student "stretched out" a square into a rectangle and "stretched up" a rectangle into a square! I can see absolutely no reason to believe that such operating is limited to only a very few students. That Pat observed it so infrequently seems to confirm my belief that more researchers need to have the creation of learning environments that are favorable to students construction of operative mathematical knowledge as a goal for their studies. My appeal here is for researchers in mathematics education to act as agents of change by specifying the constituent operations that are involved in mathematical knowledge and how those operations might be organized and reorganized by students in learning environments.

I cannot envision how it would be possible for a student to generate a definition of a concept without an operative basis being constructed from which the definition could be abstracted. To formulate a definition of a concept is a supreme achievement that requires the definer to be explicitly aware of the structure of the results of their operating. When it is the goal of the student to produce some organization on the current objects called "quadrilaterals", we can profitably use the field theorists concept of a **conceptual field** and what learning means in that context. Insight is achieved through a restructuring of the field through the individual's actions. Constructing a relation between rectangles and parallelograms cannot be done if each figure is taken simply as a visualized image of a perceptual item. Such re-presentation is crucial and I don't mean to denigrate its complexity or necessity. But it can be based on the products of pseudo-empirical abstraction. A more profitable conceptual field would seem to be to investigate the intersection of two pairs of parallel lines. Varying the distance between the lines of each pair as well as the position of each pair relative to each other would seem to provide a dynamic field in which relationships could be specified by the students that would seem artificial in other more static fields. Specifying such possible dynamic conceptual fields and how students can operate and make abstractions in those fields is a fundamental job of mathematics educators. For example, can we develop behavioral indications of those conceptual operations which lead to the specification of relations like "a rectangle is a parallelogram where the pairs of parallel lines meet at right angles" in the context of a possible dynamic field and can we discriminate among qualitative differences in conceptual operations? In other words, as researchers, we need a dynamic conceptual field in which we can understand the construction of mathematical concepts just as much as the students need dynamic conceptual fields in which they can construct mathematical concepts.

The comments that I have made up to this point apply to the work of Merlyn Behr and Guershon Harel as well as to the work of Dina Tirosh and Anna Graeber. However, what is so interesting about the Behr and Harel paper is their specification of possible sources of cognitive conflict in the application of procedures in information processing theory. If a concept image is understood as a record or a trace of the operations that were used in the

construction of the concept and if a procedure is viewed as an **evoked concept image** in the words of Tall & Vinner, then one can use the notion of a negative feedback loop (or a scheme) to reformulate the contribution of Behr and Harel. But there has to be a separation between the observer and actor as well because one is never sure exactly where Behr and Harel are with respect to the sources of cognitive conflict. They comment that cognitive conflict can be associated with Piaget's equilibration model. To me, the relation is more than association, it is the very constitution of cognitive conflict.

If a student **assimilates** a situation using a concept image and, as a result of the assimilation, establishes a goal but can find no procedure in the concept to reach the goal, the student would be in a state of activation with no means to neutralize the involved perturbation. Although there would be a fragment of the concept missing, I would not consider the knower to be in a state of cognitive conflict if what we mean by that term involves an awareness of two conflicting elements as pointed out by Dina and Anna. There would be an unrelieved tension--a lack of goal satisfaction that some might express as "it still bugs me!". Polya would say the student has a problem and I concur.

So, I would not categorize "missing a fragment" as a source of cognitive conflict. I also find violation of a rule better understood in another set of cloths. Dina and Anna use the same example as Merlyn and Guershon use when illustrating the principle. However, Dina and Anna report that preservice elementary school teachers divide when confronted with a problem **One bolt of silk fabric costs $15,000. What is the price of 0.75 of the bolt?"** because they believe the answer will be less than $15,000. The choice of a procedure (dividing) follows from the interpretation of the problem (not a re-interpretation) which, from my perspective, is quite appropriate. Conflict would be possible if the student observed that the result of carrying out the procedure was not the one expected because it would (hopefully) exceed $15,000.

I would like to in part reformulate the violation of a rule as **the violation of a belief.** This reformulation is quite compatible with how Dina and Anna view inconsistencies and their work could easily be placed on a solid epistemological foundation if they reinterpreted it in terms of scheme theory. More generally, mathematical procedures developed for operating are sometimes undifferentiated enough to yield results that violate a current belief. In the history of mathematics, the Pythagorean theorem is a classical example because it lead to a result that violated the belief that any two segments are commensurable.

Mathematical concepts are rich enough, even the elementary concepts, to yield an **alternate operative procedure** but the same result as the other procedure. For example, if children independently count-down-to in missing addend situations in which they had previously counted-up-to, this may lead to a perturbation and eventually to a reversible operative scheme which before had been unidirectional. This is another basic source of cognitive conflict.

The failure of a procedure to produce the expected result is also a primary source of cognitive conflict and can differ from the violation of a belief. In the violation of a belief, a result is experienced although it does violate a previously held belief. It is possible for a procedure to occur without a result being reached like those that have previously occurred for any one of several reasons. For example, a student might try to show that $f(x) = 1/x$ where $x = 0$ is not continuous at 0 but reach no result. This could lead to a revision of the students concept of continuity. Another example is where a child might count on to find the sum of 9 and 8 but lose track of counting because they cannot keep track of counting eight times. In this case the lack of a result might lead to a new reorganized concept of 8.

In matching problem components, the cognitive conflict that is explained seems to be to be one of the observer rather than the actor. In the example given by Merlyn and Guershon, if a student applies the digit string rule to 2.34 and 2.8 and produces the result that 2.34 is greater than 2.8, there is no internal inconsistency in this assimilation that is apparent to me. In short, "matching problem components" is a beautiful example of assimilation which is the integration of any sort of reality into a mathematical conceptual scheme in the mathematical context.

It is really interesting to me to be able to reformulate the sources of cognitive conflict in terms of sources of perturbations in scheme theory. It is every bit as interesting that Dina and Anna have distinguished between acquiring and restructuring knowledge. I take what they mean by acquiring knowledge as what has been already discussed. Restructuring knowledge as they point out is more profound and involves reflective abstraction in a way that is somewhat different than is implied by modification of a mathematical concept in the context of its immediate use. We should be alert to reorganizations that occur in no particular context as well as those that occur in a particular context.

Finally, we should not reify cognitive conflict and take it as the sole source of cognitive change. The perturbations that activate the implementation of a mathematical concept can lead to cognitive conflict. But they can also activate a successful use of the mathematical concept where there is no change required for its use or implied by its use. At last, we have those situations where elements are observed to be disregarded by the actor in assimilation. An example is given by Dina and Anna when their preservice elementary school teachers reversed the role of the divisor and the dividend when writing an expression to solve a word problem, where the former was greater than the latter. In these cases, we have a choice as an observer. We can try to create situations that might lead the actor to observe an internal inconsistency in the results of operating and ask for a resolution of the inconsistency. If we believe that a reorganization of the mathematical concept is required before the resolution is possible, we are simply asking the actor to do what is currently not possible--i.e., reorganize their concepts on the spot, so to speak. Our students would be far better off in these cases if we as teachers simply ignored the apparent misconception and proceeded on as if the students were successful, which indeed they were from their point of view. As teachers, we have to work indirectly on such reorganizations, not directly.

I have enjoyed discussing the papers and the problems they presented to me. Hopefully, what I have said will contribute to understanding possible roles of cognitive conflict in mathematics learning and teaching.

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